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S-PRIMAL IDEALS IN TRIVIAL EXTENSION AND IN AMALGAMATION ALGEBRA

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ABSTRACT. Let A be a commutative ring with identity and S be a multiplicative subset of A. In this paper, we introduce and study the notion of S-primal ideals as a generalization of the notion of primal ideals. We define an ideal I disjoint with S to be S-primal if there exists $s \in S$ such that $(I : s)$ is primal. Several properties of S-primal ideals are given. We investigate the behavior of the S-primal ideals under passage to some algebraic constructions such as homomorphic image, direct product and localization. Also, we study the relationship between the S-primal ideals, S-irreducible and S-primary ideals. Moreover, we examine the transfer of S-primal property to some ideals of trivial ring extensions and amalgamation of rings along an ideal.

1. Introduction

Let A be a commutative ring with identity. A subset S of A is called a multiplicative subset if it satisfies the following conditions: (i) $1 \in S$ and $0 \notin S$; (ii) for each $s_1, s_2 \in S$, we have $s_1s_2 \in S$. Also, we denote by $Z(A)$ the set of zero-divisors of A and if I is a proper ideal of A then, $\sqrt{I}$ denotes the radical of I. Any undefined notation or terminology is standard as in [5].

Motivated by Noether's work on the existence and uniqueness of the decomposition of an ideal in a Noetherian ring as an intersection of primary ideals, in [12], L. Fuchs introduced and studied the notion of primal ideals. An element $x \in A$ is called prime to a proper ideal I if $(I : x) = I$. Recall from [12] that a proper ideal I is said to be primal if I^* the set of elements of A which are not prime to I is an ideal of A; this ideal is then a prime and is called the adjoint ideal of I. Evidently, I^* corresponds to the set of zero-divisors in the factor ring A/I. Also in [12], the author presents several properties of primal ideals including their advantage over primary ideals such that, without any chain condition, every irreducible ideal is primal and every ideal is the intersection of primal ideals.

As a generalization of a primal ideals, considering N be a proper submodule of an A-module E. Recall from [9] that an element $a \in A$ is called prime to N if $(N \underset{E}{:} a) = N$. A proper submodule N of E is said to be primal

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if $adj(N)$; the set of all elements of A that are not prime to N, forms an ideal of A who is called the adjoint ideal of N. Note that if N is a primal submodule of E, then the ideal $adj(N)$ is a prime ideal (see [4]).

In [14], A. Hamed and A. Malek introduced and studied the concept of S-prime ideals as a generalization of prime ideals. An ideal I of A disjoint with S is said to be an S-prime ideal of A if there exists $s \in S$ such that for all $a, b \in A$ with $ab \in I$, we have either $sa \in I$ or $sb \in I$. Thus, an ideal I of A disjoint with S is S-prime if and only if there exists $s \in S$ such $(I : s)$ is a prime ideal of A.

Motivated by this generalization of prime ideals, we introduce and study the notion of S-primal ideals. Let S be a multiplicative subset of A and I be an ideal of A disjoint with S. The ideal I is called S-primal if there exists $s \in S$ such that $(I : s)^*$; the set of all elements of A which are not prime to $(I : s)$ forms an ideal of A called the s-adjoint of I. Thus, an ideal I of A disjoint with S is S-primal if and only if there exists $s \in S$ such that $(I : s)$ is primal ideal of A.

In the first section, we study some basic properties of the class of S-primal ideals. Thus, we investigate the behavior of S-primal property under homomorphic image, factor ring, direct product and localization.

The second section is devoted to the study of the relation between the concepts of S-primal, S-primary and S-irreducible ideals. It is shown that every S-irreducible ideal is S-primal and we give a necessary and sufficient condition for an S-primal ideal to be an S-primary ideal. Also, we characterize a ring in which every ideal is S-primal.

The last section deals to study the S-primal property in trivial ring extension and amalgamated ring. Also, we give examples of the ideals exhibiting this property.

2. Basic results

We recall our key definition :

Definition 2.1. *The ideal I is called S-primal if there exists $s \in S$ such that $(I : s)^*$; the set of all elements of A which are not prime to $(I : s)$ forms an ideal of A called the s-adjoint of I.*

Remark 2.2.

1. *If S consists of units of a ring A, then the primal and the S-primal ideals coincide.*
2. *Every primal ideal is S-primal. The converse need not true in general. Indeed, we consider $A = \mathbb{Z}$, $S = \{2^n / n \in \mathbb{N}\}$ and $I = 6\mathbb{Z}$. Clearly $I \cap S = \emptyset$ and I is S-primal since $(I : 2)^* = (3)^* = (3)$. But, I is not primal ideal of A since $2 \in I^*$ and $3 \in I^*$ but $1 = 3 - 2 \notin I^*$.*
3. *If $S \cap I^* = \emptyset$ then, $(I : s)^* = I^*$ for all $s \in S$, and thus the notions of primal and the S-primal ideals coincide.*

Let $f : A \longrightarrow B$ be a ring homomorphism. Recall that if $f(S)$ does not contain zero, then it is a multiplicative subset of B. Conversely, if T is a multiplicative subset of B, then $f^{-1}(T)$ is a multiplicative subset of A.

The following result investigates the stability of S-primal property under homomorphic image.

Proposition 2.3. *Let $f : A \longrightarrow B$ be a surjective ring homomorphism and S and T are multiplicative subsets of A and B respectively. Then:*

(1) If J is a T-primal ideal of B, then $f^{-1}(J)$ is an $f^{-1}(T)$-primal ideal of A.

(2) Suppose that $Ker(f) \subseteq I$. If I is an S-primal ideal of A, then $f(I)$ is an $f(S)$-primal ideal of B.

The proof of this proposition draws on the following lemma.

Lemma 2.4. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B. Then :*

(1) $(f^{-1}(J))^* \subseteq f^{-1}(J^*)$.

(2) If f is surjective, then $(f^{-1}(J))^* = f^{-1}(J^*)$.

Proof.
(1) Let $x \in (f^{-1}(J))^*$. Then $(f^{-1}(J) : x) \not\subset f^{-1}(J)$. So there exists $a \in A$ such that $ax \in f^{-1}(J)$ but $a \notin f^{-1}(J)$. Thus $f(a)f(x) \in J$ and $f(a) \notin J$. Then $f(x) \in J^*$ and hence $x \in f^{-1}(J^*)$.
(2) Let $x \in f^{-1}(J^*)$. Then $f(x) \in J^*$ and so $(J : f(x)) \not\subset J$. Hence, there exists $b \in B$ such that $bf(x) \in J$ and $b \notin J$. Since f is surjective, then there exists $a \in A$ such that $b = f(a)$. Thus $f(ax) \in J$ and $f(a) \notin J$ and then $ax \in f^{-1}(J)$ and $a \notin f^{-1}(J)$. It follows that $x \in (f^{-1}(J))^*$. Therefore $(f^{-1}(J))^* = f^{-1}(J^*)$.

□

Proof of proposition 2.3
(1) First, note that for each $s \in T$ and $r \in f^{-1}(T)$ such that $f(r) = s$, we have $(f^{-1}(J) : r) = f^{-1}(J : s)$. Suppose that J is T-primal of B. Then there exists $t \in T$ such that $(J : t)$ is a primal ideal of B and since f is surjective, then there exists $a \in A$ such that $f(a) = t$. Since $T \cap J = \emptyset$, we get $f^{-1}(T) \cap f^{-1}(J) = \emptyset$. By the definition, $(J : t)^*$ is an ideal of B. So $f^{-1}((J : t)^*)$ is an ideal of A. As $(f^{-1}(J : t))^* = ((f^{-1}(J) : a))^*$ and $(f^{-1}(J : t))^* = f^{-1}((J : t)^*)$ by Lemma 2.4, we get that $(f^{-1}(J) : a)$ is a primal ideal of A, where $a \in f^{-1}(T)$. Therefore $f^{-1}(J)$ is an $f^{-1}(T)$-primal ideal of A.
(2) Assume that I is an S-primal ideal of A. Then there exists $s \in S$ such that $(I : s)^*$ is an ideal of A. Since $Ker(f) \subseteq I$ and $I \cap S = \emptyset$, then $0 \notin f(S)$ and $f(I) \cap f(S) = \emptyset$. Let $y_1, y_2 \in (f(I) : f(s))^*$. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in A$, and $((f(I) : f(s)) : f(x_1)) \not\subset (f(I) : f(s))$ and $((f(I) : f(s)) : f(x_2)) \not\subset (f(I) : f(s))$. So it is easy to deduce that $f(I : sx_1) \not\subset f(I : s)$ and $f(I : sx_2) \not\subset f(I : s)$. Then $(I : sx_1) \not\subset (I : s)$ and $(I : sx_2) \not\subset (I : s)$. Thus $x_1, x_2 \in (I : s)^*$ and then $((I : s) : x_1{+}x_2) \not\subset (I : s)$. So, we can easily conclude $((f(I) : f(s)) : y_1 + y_2) = f((I : s) : x_1 + x_2) \not\subset$

$(f(I) : f(s))$. Therefore, $y_1 + y_2 \in (f(I) : f(s))^*$, as desired. □

When S is a subset of units of A, we obtain the following result.

Corollary 2.5. *Let $f : A \longrightarrow B$ be a surjective ring homomorphism. Then*

(1) *If J is a primal ideal of B, then $f^{-1}(J)$ is a primal ideal of A.*
(2) *Suppose that $Ker(f) \subseteq I$. If I is a primal ideal of A, then $f(I)$ is a primal ideal of B.*

Remark 2.6. *It is known that the contraction of a primary ideal by a ring homomorphism is a primary ideal. However, this is not the case for a primal ideal. You can see the Example 3.7 which is announced next.*

It is easy to check that $\bar{S} = \{\bar{s} = s + I/s \in S\}$ is a multiplicative subset of A/I since $I \cap S = \emptyset$.

The following proposition establishes a bijective correspondence between the set of S-primal ideals of A containing I and the set of $\bar{S}$-primal ideals of A/I when $\bar{S}$ is a multiplicative subset of A/I.

Proposition 2.7. *Let J be an ideal of A containing I. The map $J \mapsto J/I$ sets up a bijective correspondence between the set of S-primal ideals of A containing I and the set of $\bar{S}$-primal ideals of A/I.*

Proof. Suppose that J is an S-primal ideal of A containing I. Let $\varphi : A \longrightarrow A/I$ be the canonical surjection. By Proposition 2.3, $J/I = \varphi(J)$ is $\bar{S}$-primal ideal of A/I. Conversely, let K be an $\bar{S}$-primal ideal of A/I. Then $K = J/I$, where J is an ideal of A containing I. It follows from Proposition 2.3 that $J = \varphi^{-1}(K)$ is an $\varphi^{-1}(\bar{S})$-primal ideal of A. Since $\varphi^{-1}(\bar{S}) = I + S$ and $J \cap (I + S) = \emptyset$, then $J \cap S = \emptyset$. Also, there exist $s \in S$ and $i \in I$ such that $(J : i + s)$ is a primal ideal of A. Since $(J : i + s) = (J : s)$ (because $i \in J$) then $(J : s)$ is a primal ideal of A. Thus J is an S-primal ideal of A. □

Our next proposition characterizes S-primal ideals in a finite direct product of rings. To do this, we recall that if S_i is a multiplicative subset of a ring A_i for all $i \in \{1, ..., n\}$, then $S_1 \times S_2 \times \cdots \times S_n$ is a multiplicative subset of the ring $A_1 \times A_2 \times \cdots \times A_n$.

Proposition 2.8. *Let $S_1, S_2, ..., S_n$ be multiplicative subsets of rings $A_1, A_2, ..., A_n$ respectively. Then $I_1 \times I_2 \times \cdots \times I_n$ is an $(S_1 \times S_2 \times \cdots \times S_n)$-primal ideal of $A_1 \times A_2 \times ... \times A_n$ if and only if there exists i such that I_i is an S_i-primal ideal of A_i and for each $j \neq i$ we have $S_j \cap I_j \neq \emptyset$.*

Proof of proposition 2.8 (1) Without loss of generality, we may assume that $n = 2$. Suppose that $I_1 \times I_2$ is an $(S_1 \times S_2)$-primal ideal of $A_1 \times A_2$. Since $(I_1 \times I_2) \cap (S_1 \times S_2) = \emptyset$, then there exists $i \in \{1, 2\}$ such that $I_i \cap S_i = \emptyset$. We may assume that $I_1 \cap S_1 = \emptyset$. Since $I_1 \times I_2$ is an $(S_1 \times S_2)$-primal ideal of $A_1 \times A_2$, then there exists $(s_1, s_2) \in S_1 \times S_2$ such that $((I_1 \times I_2) : (s_1, s_2))^* = ((I_1 : s_1) \times (I_2 : s_2))^*$ is an ideal of $A_1 \times A_2$. Suppose that $s_2 \notin I_2$, then $(1,0)(0,1) \in (I_1 : s_1) \times (I_2 : s_2)$ but $(1,0) \notin (I_1 : s_1) \times (I_2 : s_2)$ (since $s_1 \notin I_1$) and hence $(0,1) \in ((I_1 : s_1) \times (I_2 : s_2))^*$. Moreover $(0,1)(1,0) \in (I_1 : s_1) \times (I_2 : s_2)$ but $(0,1) \notin (I_1 : s_1) \times (I_2 : s_2)$ (since $s_2 \notin I_2$) and hence $(1,0) \in ((I_1 : s_1) \times (I_2 : s_2))^*$. Then $(1,1) = (1,0)+(0,1) \in ((I_1 : s_1) \times (I_2 : s_2))^*$, which is absurd. Thus we have $s_2 \in I_2$ and then $S_2 \cap I_2 \neq \emptyset$. So we will have that $((I_1 : s_1) \times A_2)^*$ is an ideal of $A_1 \times A_2$ since $(I_2 : s_2) = A_2$. We show that $((I_1 : s_1) \times A_2)^* = (I_1 : s_1)^* \times A_2$, indeed $(x, y) \in ((I_1 : s_1) \times A_2)^*$ if and only if there exists $(a, b) \in A_1 \times A_2$ such that $(a, b) \notin (I_1 : s_1) \times A_2$ and $(a, b)(x, y) \in (I_1 : s_1) \times A_2$ if and only if there exists $(a, b) \in A_1 \times A_2$ such that $a \notin (I_1 : s_1)$ and $ax \in (I_1 : s_1)$ if and only if $(x, y) \in (I_1 : s_1)^* \times A_2$. Thus, $(I_1 : s_1)^*$ is an ideal of A_1 which implies that I_1 is an S_1-primal ideal of A_1. Conversely, assume that I_1 is an S_1-primal ideal of A_1 and $S_2 \cap I_2 \neq \emptyset$. Then $(I_1 \times I_2) \cap (S_1 \times S_2) = \emptyset$ and there exists $s_1 \in S_1$ such that $(I_1 : s_1)^*$ is an ideal of A_1. Let $s_2 \in S_2 \cap I_2$, then $(I_2 : s_2) = A_2$ and so $((I_1 \times I_2) : (s_1, s_2)))^* = ((I_1 : s_1) \times A_2)^* = (I_1 : s_1)^* \times A_2$ is an ideal of $A_1 \times A_2$, as desired. □

When $S = \{1\}$, we get the same result found in [11, Lemma 13].

Corollary 2.9. *$I_1 \times I_2 \times \cdots \times I_n$ is a primal ideal of $A_1 \times A_2 \times ... \times A_n$ if and only if there exists i such that I_i is a primal ideal of A_i and for each $j \neq i$ we have $I_j = A_j$.*

Now, we are going to look at the behavior of the S-primal ideals under passage to localization. Let $\phi : A \to S^{-1}A$ be the canonical map. It is well known that since $0 \notin \phi(S)$, then $\phi(S)$ is a multiplicative subset of $S^{-1}A$.

Proposition 2.10. *Under the above notation, assume that $S \cap I^* = \emptyset$. Let T be a multiplicative subset of A. If I is a T-primal ideal of A, then $S^{-1}I$ is $\phi(T)$-primal of $S^{-1}A$.*

Proof. Assume that I is T-primal. Since $S \cap I^* = \emptyset$, then $\phi^{-1}(S^{-1}I) = I$ and $0 \notin \phi(T)$. Note that $S^{-1}I \cap \phi(T) = \emptyset$, otherwise, there exists $t \in T$ such that $\frac{t}{1} \in S^{-1}I$ and then $t \in I \cap T = \emptyset$, absurd. Also, there exists $t \in T$ such that $(I : t)$ is a primal ideal of A. We show that $S^{-1}((I : t)^*) = (S^{-1}I : \frac{t}{1})^*$. Indeed, let $\frac{x}{s} \in S^{-1}((I : t)^*)$. Then, there exists $u \in S$ such that $ux \in (I : t)^*$. Thus $((I : u) : xt) \not\subset (I : t)$. Since $u \notin I^*$ then $(I : u) = I$ and so $(I : xt) \not\subset (I : t)$. Hence, there exists $a \in A$ such that $axt \in I$ and $at \notin I$. Thus $\frac{at}{1} \notin S^{-1}I$ and $\frac{axt}{s} \in S^{-1}I$. So $\frac{a}{1} \notin (S^{-1}I : \frac{t}{1})$ and $\frac{a}{1}\frac{x}{s} \in (S^{-1}I : \frac{t}{1})$. Therefore $\frac{x}{s} \in (S^{-1}I, \frac{t}{1})^*$. Conversely, let $\frac{x}{s} \in (S^{-1}I : \frac{t}{1})^*$, then there exists $\frac{a}{u} \in S^{-1}A$ such that $\frac{axt}{us} \in S^{-1}I$ and $\frac{at}{u} \notin S^{-1}I$. So $axt \in I$ and $at \notin I$ and then $ax \in (I : t)$ and $a \notin (I : t)$. Thus $x \in (I : t)^*$ and then $\frac{x}{s} \in S^{-1}((I : t)^*)$. Since $(I : t)$ is a primal ideal of A, then $(I : t)^*$ is an ideal of A. So $(S^{-1}I : \frac{t}{1})^* = S^{-1}(I : t)^*$ is an ideal of $S^{-1}A$. Consequently, $S^{-1}I$ is a $\phi(T)$-primal ideal of $S^{-1}A$. □

We study the relationship between the S-primal, S-irreducible and S-primary ideals. We show that an S-irreducible ideal is S-primal and we give necessary and sufficient conditions for an S-primal ideal to be S-primary. Also, we characterize the rings where every proper ideal is primal.

We start by generalizing of [12, Theorem 1]. We say that the ideal I disjoint with S is S-irreducible if there exists $s \in S$ such that $(I : s)$ is irreducible.

Proposition 2.11. *If I is an S-irreducible ideal of A, then I is an S-primal ideal of A.*

Proof. Suppose that I is not S-primal ideal of A and let $s \in S$. Then $(I : s)^*$ is not an ideal of A. Thus there exist $x, y \in (I : s)^*$ such that $x + y \notin (I : s)^*$. So $(I : s) = (I : sx + sy) \supseteq (I : sx) \cap (I : sy)$. Also, we have $(I : s) \subseteq (I : sx) \cap (I : sy)$. Then $(I : s) = (I : sx) \cap (I : sy)$. Since $(I : sx) \neq (I : s)$ and $(I : sy) \neq (I : s)$, we get $(I : s)$ is not irreducible and hence I is not S-irreducible. $\square$

The following example shows that the converse of the previous proposition need not true in general.

Example 2.12. *Let $A = K[x, y, z]$, where K is a field and let $S = \{z^n, n \in \mathbb{N}\}$. Let $I = (x, y)^2 = (x^2, xy, y^2)$. First, note that $I \cap S = \emptyset$. By* [5, Exercice 8, P: 55]*, I is a primary. Then I is a primal and so I is an S-primal ideal of A. But, for $n \geq 1$, $(I : z^n) = \{0\}$ and $(I : 1)$ $(= I = (x^2, y) \cap (x, y^2))$ are not irreducible ideals of A. Hence, I is not S-irreducible ideal of A.*

Recall from [18], that an ideal I of A disjoint with S is S-primary if there exists $s \in S$ such that for all $x, y \in A$, if $xy \in I$, then $sx \in I$ or $sy \in \sqrt{I}$. The following result specifies the relationship between S-primary and S-primal ideals.

Proposition 2.13. *Let I be an ideal of A. Then I is S-primary if and only if I is S-primal and there exists $s \in S$ such that $\sqrt{(I : s)} = (I : s)^*$.*

Proof. (1) We claim that for each $s \in S$, we have $\sqrt{(I : s)} \subseteq (I : s)^*$. Indeed, let $x \in \sqrt{(I : s)}$ and $n \in \mathbb{N}^*$ the smallest integer such that $x^n \in (I : s)$. So $x^{n-1}x \in (I : s)$ but $x^{n-1} \notin (I : s)$. Hence $x \in (I : s)^*$. Now, assume that I is S-primary. Then there exists $s \in S$ such that $(I : s)$ is primary. Let $x \in (I : s)^*$. Then $(I : sx) \not\subseteq (I : s)$. So there exists $a \in A$ such that $ax \in (I : s)$ and $a \notin (I : s)$. Since $(I : s)$ is primary, we get $x \in \sqrt{(I : s)}$. Thus $\sqrt{(I : s)} = (I : s)^*$. Therefore $(I : s)^*$ is an ideal of A and then I is S-primal. Conversely, suppose that $\sqrt{(I : s)} = (I : s)^*$ for some $s \in S$. Let $x, y \in A$ such that $xy \in I$ and $xs \notin I$. Then $xy \in (I : s)$ and $x \notin (I : s)$. So $y \in (I : s)^* = \sqrt{(I : s)}$. Hence, there exists $n \in \mathbb{N}^*$ such that $y^n s \in I$ and then $ys \in \sqrt{I}$. $\square$

In the last part of this section, we characterizes rings in which every ideal disjoint with S is S-primal and we will deduce a characterization of the rings where any proper ideal is primal. To do this, we need the following lemma.

Lemma 2.14. *Let I, J be ideals of A.*

(1) I is a prime ideal of A if and only if $I = I^*$.
(2) If $I, J \in Spec(A)$, $I \nsubseteq J$ and $J \nsubseteq I$, then $(I \cap J)^* = I \cup J$.

Proof.
(1) Assume that I is a prime ideal of A and let $x \in I^*$. So $(I : x) \nsubseteq I$ and then there exists $a \in A$ such that $ax \in I$ and $a \notin I$. Thus $x \in I$. The reverse implication is obvious.
(2) Let $I, J \in Spec(A)$ and $x \in (I \cap J)^*$, then $(I \cap J : x) \nsubseteq I \cap J$. So $(I : x) \cap (J : x) \nsubseteq I \cap J$ and so either $(I : x) \nsubseteq I$ or $(J : x) \nsubseteq J$. Thus $x \in I^* = I$ or $x \in J^* = J$. Conversely, let $x \in I = I^*$, then there exists $a \in A$ such that $ax \in I$ and $a \notin I$. Let $j \in J \setminus I$, we have $ajx \in I \cap J$ and $aj \notin I \cap J$ since $aj \notin I$. So $x \in (I \cap J)^*$ and then $I \subseteq (I \cap J)^*$. In the same way we show that $J \subseteq (I \cap J)^*$. Thus $(I \cap J)^* = I \cup J$.

□

We recall that an ideal I of A disjoint with S is said to be an S-prime ideal if and only if $(I : s)$ is prime for some $s \in S$ [14, Proposition 1]. In this case, $(I : s)$ is a prime ideal corresponding to the ideal I. Let $\mathfrak{P}$ the set of all prime ideals corresponding to all S-prime ideals, i.e., $\mathfrak{P} = \{(I : s)/I$ is an ideal of A disjoint with S, $s \in S$, and $(I : s)$ is prime$\}$.

Theorem 2.15. *If every ideal of A disjoint with S is S-primal, then $\mathfrak{P}$ is totally ordered. The converse is hold if there exists $s \in S$ such that $(I : s)^*$ is disjoint with S.*

Proof. Assume that every ideal of A disjoint with S is S-primal. Suppose that $\mathfrak{P}$ is not totally ordered. Then there exist $(I : s), (J : t) \in \mathfrak{P}$ such that $(I : s) \nsubseteq (J : t)$ and $(J : t) \nsubseteq (I : s)$. Since $(I \cap J : st)$ is disjoint with S, then $(I \cap J : st)$ is S-primal so there exists $u \in S$ such that $(I \cap J : stu)^*$ is an ideal of A. On the other hand, since $(I : s)$ and $(J : t)$ are prime, it follows from the previous lemma that $(I : s) = (I : s)^*$ and $(J : t)^* = (J : t)$. Also, since I and J are disjoint with S, then $tu \notin (I : s) = (I : s)^*$ and $su \notin (J : t) = (J : t)^*$. Hence, $(I : stu) = (I : s)$ and $(J : tsu) = (J : t)$ thus $(I \cap J : stu) = (I : stu) \cap (J : stu) = (I : s) \cap (J : t)$. Then, by the previous lemma, we obtain $(I \cap J : stu)^* = (I : s) \cup (J : t)$ is an ideal, absurd. Conversely, we show under the hypothesis that $(I : s)^*$ is disjoint with S for some $s \in S$ that every ideal of A is S-primal. Let I be an ideal of A disjoint with S. Then there exists $s \in S$ such that $(I : s)^* \cap S = \emptyset$. As in [16, theorem 2], $A \setminus (I : s)^*$ is a saturated multiplicative subset of A and so $(I : s)^* = \cup_{i \in I} P_i$, where P_i is a prime ideal of A for all $i \in I$. Since $(I : s)^* \cap S = \emptyset$, then $P_i \cap S = \emptyset$ for every i. So, $P_i = (P_i : 1) \in \mathfrak{P}$. Therefore $(P_i)_{i \in I}$ is a chain and then $\cup_{i \in I} P_i$ is an ideal of A. Thus $(I : s)^*$ is an ideal of A, as desired. □

The following corollary characterizes rings in which every proper ideal is primal.

Corollary 2.16. *Every proper ideal of A is a primal if and only if $Spec(A)$ is totally ordered.*

Proof. It suffices to consider a multiplicative subset which consist of units elements and to apply the previous Theorem.

□

Corollary 2.17. *(1) If A is a chained ring then, every proper ideal of A is primal.*
(2) If A is a valuation ring then, every proper ideal of A is primal.

3. S-primal in trivial extension and amalgamation algebra

In this section, we study the S-primal property in Nagata's idealization. To do this, we recall some facts on the idealization. The idealization of E in A (or trivial extension of A by E) is a commutative ring

$$A \propto E = \{(a,e)/a \in A, e \in E\}$$

under the usual addition and the multiplication defined as $(a,e)(b,f) = (ab, af + be)$ for all (a,e), $(b,f) \in A \propto E$ (see for instance [1],[3],[10]). It was shown in [1] that if S is a multiplicative subset of A, then $S \propto 0$ is a multiplicative subset of $A \propto E$. Note that, an ideal I is disjoint with a multiplicative subset S of A if and only if $I \propto E$ is disjoint with $S \propto 0$.

Theorem 3.1. *Let A be a ring, S be a multiplicative subset of A, E be an A-module and let I be a proper ideal of A. Then :*

(1) $I \propto E$ is an $S \propto 0$-primal ideal of $A \propto E$ if and only if I is an S-primal ideal of A.
(2) Let N be a proper submodule of E such that $IE \subseteq N$ and assume that $adj(N) \cap S = \emptyset$. Then $I \propto N$ is an $S \propto 0$-primal ideal of $A \propto E$ if and only if there exists $s \in S$ such that $(I : s)^* \cup adj(N)$ is an ideal of A.

Proof. (1) It remains to show that $(I \propto E : (s,0))^* = (I : s)^* \propto E$, where $s \in S$. Indeed, let $(x,f) \in (I \propto E : (s,0))^*$. There exists $(b,g) \notin (I \propto E : (s,0))$ such that $(b,g)(x,f) \in (I \propto E : (s,0))$. So $(b,g)(s,0) \notin I \propto E$ and $(b,g)(x,f)(s,0) \in I \propto E$. Thus $b \notin (I : s)$ and $bx \in (I : s)$ and then $x \in (I : s)^*$. Therefore $(x,f) \in (I : s)^* \propto E$. Conversely, let $(x,f) \in (I : s)^* \propto E$, then $x \in (I : s)^*$. So there exists $a \in A$ such that $as \notin I$ and $axs \in I$. Thus $(a,0)(s,0) \notin I \propto E$ and $(a,0)(x,f)(s,0) \in I \propto E$. Then $(a,0) \notin (I \propto E : (s,0))$ and $(a,0)(x,f) \in (I \propto E : (s,0))$. Thus $(x,f) \in (I \propto E : (s,0))^*$. Finally, the result follows from the fact that, $(I \propto E : (s,0))^*$ is an ideal of $A \propto E$ if and only if $(I : s)^*$ is an ideal of A.

(2) Under additional hypothesis $adj(N) \cap S = \emptyset$, we prove that $(I \propto N : (s,0))^* = ((I : s)^* \cup adj(N)) \propto E$. Let $(x,e) \in (I \propto N : (s,0))^*$, then there exists $(b,y) \in A \propto E$ such that $(b,y) \notin (I \propto N : (s,0))$ and $(b,y)(x,e) \in (I \propto N : (s,0))$. So $(b,y)(s,0) = (bs, ys) \notin I \propto N$ and $(b,y)(x,e)(s,0) = (bxs, bse + ysx) \in I \propto N$. Then $bxs \in I$, $bse + ysx \in N$, and either $bs \notin I$ or $ys \notin N$. Two cases are then possible:
Case 1: If $bs \notin I$, then $b \notin (I : s)$ and $bx \in (I : s)$. Thus $x \in (I : s)^*$ and then $(x,e) \in ((I : s)^* \cup adj(N)) \propto E$.
Case 2: If $bs \in I$, then $ys \notin N$. Since $bse + ysx \in N$ and $IE \subseteq N$, then $bse \in N$ and so $ysx \in N$. Hence $x \in adj(N)$ and then $(x,e) \in ((I : s)^* \cup adj(N)) \propto E$.
From Cases 1 and 2, $(I \propto N : (s,0))^* \subseteq ((I : s)^* \cup adj(N)) \propto E$. Now, let $(x,e) \in ((I : s)^* \cup adj(N)) \propto E$, then $x \in (I : s)^* \cup adj(N)$. So, if $x \in (I : s)^*$, then there exists $b \in A$ such that $bx \in (I : s)$ and $b \notin (I : s)$. Thus $bxs \in I$ and $bs \notin I$. If $bE \subseteq N$, then $(x,e)(b,0)(s,0) = (bxs, bes) \in I \propto N$ and $(b,0)(s,0) = (bs,0) \notin I \propto N$. Hence $(x,e)(b,0) \in (I \propto N : (s,0))$ and $(b,0) \notin (I \propto N : (s,0))$. Therefore $(x,e) \in (I \propto N : (s,0))^*$. Now, if $bE \not\subseteq N$, then there exists $f \in E$ such that $bf \notin N$. So, $(x,e)(0,bf)(s,0) = (0, xbfs) \in I \propto N$ since $bxs \in I$ and $IE \subseteq N$. Also, $(0,bf)(s,0) = (0,bsf) \notin I \propto N$ (otherwise, $bfs \in N$ and $bf \notin N$ implies that $s \in adj(N) \cap S$, absurd). Hence $(x,e)(0,bf) \in (I \propto N : (s,0))$ and $(0,bf) \notin (I \propto N : (s,0))$. Therefore $(x,e) \in (I \propto N : (s,0))^*$. On the other hand, if $x \in adj(N)$, then there exists $m \notin N$ such that $xm \in N$. Then $(x,e)(0,m)(s,0) = (0, xms) \in I \propto N$ and $(0,m)(s,0) = (0,ms) \notin I \propto N$ ($ms \notin N$, otherwise $s \in adj(N)$, absurd). Thus $(x,e)(0,m) \in (I \propto N : (s,0))$ and $(0,m) \notin (I \propto N : (s,0))$ which means that $(x,e) \in (I \propto N : (s,0))^*$. Therefore $((I : s)^* \cup adj(N)) \propto E \subseteq (I \propto N : (s,0))^*$, as desired.

□

Remark 3.2. *(1) $(I \propto E : (s,e))^* = (I : s)^* \propto E$, for all $s \in S$ and $e \in E$.*

(2) Let N be a proper submodule of E such that $IE \subseteq N$ and $adj(N) \cap S = \emptyset$. Then $(I \propto N : (s,0))^* = ((I : s)^* \cup adj(N)) \propto E$, for all $s \in S$.

As a consequence of Theorem 3.1 in the case when S is a subset of units, we announce the following result.

Corollary 3.3. *(1) $I \propto E$ is a primal ideal of $A \propto E$ if and only if I is a primal ideal of A. In this case $(I \propto E)^* = I^* \propto E$.*

(2) Let N be a proper submodule of E such that $IE \subseteq N$. Then $I \propto N$ is a primal ideal of $A \propto E$ if and only if $I^* \cup adj(N)$ is an ideal of A. In this case $(I \propto N)^* = (I^* \cup adj(N)) \propto E$.

The following examples show that $I \propto N$ may not be primal even if I is primal ideal of A or N is primal submodule of E.

Example 3.4. *(1) Let $A = E = \mathbb{Z}, I = \{0\}$, and $N = 6\mathbb{Z}$. Then $I \propto N$ is an ideal of $A \propto E$, I is a primal ideal of A, and $I^* = I \subsetneq adj(N)$. But, $I \propto N$ is not primal since $(I \propto N)^* = adj(N) \propto \mathbb{Z} = (2\mathbb{Z} \cup 3\mathbb{Z}) \propto \mathbb{Z}$ is not an ideal of $\mathbb{Z} \propto \mathbb{Z}$.*

(2) Let $A = E = K \times K$, where K is a field, $I = \{(0,0)\}$ and $N = \{0\} \times K$. Then $I \propto N$ is an ideal of $A \propto E$, N is a primal submodule of E ($adj(N) = N$), and $adj(N) \subseteq I^* = (\{0\} \times K) \cup (K \times \{0\})$. But, $I \propto N$ is not primal of $A \propto E$ since $(\{0\} \times K) \cup (K \times \{0\})$ is not an ideal of A.

There are in general primal ideals which are not primary; as shown by ([12], page 2). In the following example, we give an S-primal ideal that is not S-primary :

Example 3.5. *Let $A = E = \mathbb{Z}[x, y]$ and $S = \{(2^n, 0)/\ n \in \mathbb{N}\}$. Since (xy, y^2) is S-primal, then $K = (xy, y^2) \propto \mathbb{Z}[x, y]$ is an S-primal ideal of $A \propto E$ by Theorem 3.1, but K is not S-primary ideal. Otherwise, since $(xy, 0) = (x, 0)(y, 0) \in K$ then, there exists $n \in \mathbb{N}$ such that $2^n y \in (xy, y^2)$ or $2^{nk}x^k \in (xy, y^2)$ for some $k \in \mathbb{N}^*$, absurd.*

Proposition 3.6. *Let N be a primal submodule of E such that $IE \subseteq N$. Then, if I is an S-primary ideal of A and $adj(N) \cap S = \emptyset$, then $I \propto N$ is an $S \propto 0$-primal ideal of $A \propto E$.*

Proof.
(1) Since I is an S-primary ideal of A, then by Proposition 2.13 there exists $s \in S$ such that $(I : s)^* = \sqrt{(I : s)}$. We show that $(I : s)^* \subseteq adj(N)$. Indeed, let $a \in (I : s)^*$ so there exists an integer $n \geq 1$ such that $a^n \in (I : s)$. Let $m \in E \setminus N$ then $a^n s \in I$ and $a^n sm \in N$. Then, $a^n m \in (N : s) = N$ (since $s \notin adj(N)$). Therefore $a^n \in adj(N)$ and hence $a \in adj(N)$. Thus, $(I : s)^* \cup adj(N) = adj(N)$ is an ideal of A and then $I \propto N$ is an $S \propto 0$-primal ideal of $A \propto E$ by Theorem 3.1. □

In general, the contraction of a primal ideal by a ring homomorphism is not primal as shown in the following example:

Example 3.7. *Consider the ring $A = \mathbb{Z}[X]$, the A-module $E = \mathbb{Z}[X]$, and the ring homomorphism $f : A \to A \propto E$, such that for each $a \in A$, we have $f(a) = (a, 0)$. Let $I = (2X)$ be an ideal of A and $N = (4, 2X, X^2)$ be a submodule of E. Since $I^* = 2Z[X] \cup XZ[X]$ and $adj(N) = (2, X)$, then by Corollary 3.3 $I \propto N$ is a primal ideal of $A \propto E$. However, $f^{-1}(I \propto N) = I$ is not primal ideal of A (since I^* is not an ideal of A).*

Now, we investigate the transfer of S-primal property to some ideals of amalgamation of rings. Let A and B be two rings, J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j)/a \in A, j \in J\}$$

is called the *amalgamation of A and B along J with respect to f*. The concept of amalgamation is an important and an interesting concept that received a considerable attention by well-known established algebraists. Motivations and additional applications of the amalgamations are discussed in detail in [6, 7, 8]. Let S be a multiplicative subset of A then $S \bowtie^f J = \{(s, f(s)+k)/s \in S, k \in J\}$ is a multiplicative subset of $A \bowtie^f J$, in particular $S' = \{(s, f(s))/s \in S\}(\subseteq S \bowtie^f J)$ is a multiplicative subset of $A \bowtie^f J$. If $0 \notin f(S)$ then $f(S)$ is a multiplicative subset of $f(A)+J$. Also, if $f(S) \cap J = \emptyset$ then $f(S)+J$ is a multiplicative subset of $f(A)+J$.
Note that if L is an ideal of $f(A)+J$, then $\bar{L}^f = \{(a, f(a)+k)/a \in A, k \in J, f(a)+k \in L\}$ is an ideal of $A \bowtie^f J$.
Throughout this last part, A and B will be two rings, $f : A \longrightarrow B$ be a ring homomorphism, and S be a multiplicative subset of A. I will be a proper ideal of A and J, L be ideals of B and $f(A)+J$ respectively.

Theorem 3.8. *We preserve the above notation.*

(1) The following statements are equivalent:
  (a) *$I \bowtie^f J$ is an S'-primal ideal of $A \bowtie^f J$.*
  (b) *$I \bowtie^f J$ is an $(S \bowtie^f J)$-primal ideal of $A \bowtie^f J$.*
  (c) *I is an S-primal ideal of A.*

(2) Assume that $f(S) \cap J = \emptyset$. Then $\bar{L}^f$ is an $(S \bowtie^f J)$-primal ideal of $A \bowtie^f J$ if and only if L is an $(f(S)+J)$-primal ideal of $f(A)+J$.

The proof of Theorem 3.8 draws on the following lemma.

Lemma 3.9. *(1) $(I \bowtie^f J : (s, f(s)+k)) = (I : s) \bowtie^f J$, where $s \in S$ and $k \in J$.*

(2) $((I : s) \bowtie^f J)^* = (I : s)^* \bowtie^f J$, where $s \in S$.

(3) $(\bar{L}^f : (s, f(s)+k))^*$ is an ideal of $A \bowtie^f J$ if and only if $(L : f(s)+k)^*$ is an ideal of $f(A)+J$, where $s \in S$ and $k \in J$.

Proof.
(1) Let $(s, f(s)+k) \in S \bowtie^f J$ and $(x, f(x)+j) \in A \bowtie^f J$. Then $(x, f(x)+j) \in (I : s) \bowtie^f J$ if and only if $xs \in I$ if and only if $(x, f(x)+j)(s, f(s)+k) = (xs, f(xs)+jf(s)+kf(x)+jk) \in I \bowtie^f J$ if and only if $(x, f(x)+j) \in (I \bowtie^f J : (s, f(s)+k))$.
(2) Let $(x, f(x)+k) \in (I : s)^* \bowtie^f J$, then $x \in (I : s)^*$. So, there exists $a \in A$ such that $ax \in (I : s)$ and $a \notin (I : s)$. Thus $(a, f(a)) \notin (I : s) \bowtie^f J$ and $(a, f(a))(x, f(x)+k) = (ax, f(ax)+f(a)k) \in (I : s) \bowtie^f J$. Then $(x, f(x)+k) \in ((I : s) \bowtie^f J)^*$. Now, let $(x, f(x)+k) \in ((I : s) \bowtie^f J)^*$. Then there exists $(a, f(a)+j) \in A \bowtie^f J$ such that $(a, f(a)+j) \notin (I : s) \bowtie^f J$ and $(a, f(a)+j)(x, f(x)+k) \in (I : s) \bowtie^f J$. So $ax \in (I : s)$ and $a \notin (I : s)$ and then $x \in (I : s)^*$. Therefore $(x, f(x)+k) \in (I : s)^* \bowtie^f J$.
(3) Assume that $(\bar{L}^f : (s, f(s)+k))^*$ is an ideal of $A \bowtie^f J$ and let $y_1 = f(a_1)+k_1, y_2 = f(a_2)+k_2 \in (L : f(s)+k)^*$. Then there exists $f(b_1)+j_1, f(b_2)+j_2 \in f(A)+J$ such that $f(b_1)+j_1, f(b_2)+j_2 \notin (L : f(s)+k)$

and $(f(a_1)+k_1)(f(b_1)+j_1),(f(a_2)+k_2)(f(b_2)+j_2) \in (L : f(s)+k)$. So $(b_1, f(b_1)+j_1),(b_2, f(b_2)+j_2), \notin (\bar{L}^f : (s, f(s)+k))$ and $(a_1, f(a_1)+k_1)(b_1, f(b_1)+j_1),(a_2, f(a_2)+k_2)(b_2, f(b_2)+j_2) \in (\bar{L}^f : (s, f(s)+k))$. Then $(a_1, f(a_1)+k_1),(a_2, f(a_2)+k_2) \in (\bar{L}^f : (s, f(s)+k))^*$. So $(a_1, f(a_1)+k_1)+(a_2, f(a_2)+k_2) \in (\bar{L}^f : (s, f(s)+k))^*$. Hence, there exists $(c, f(c)+j) \notin (\bar{L}^f : (s, f(s)+k))$ such that $(c, f(c)+j)(a_1+a_2, f(a_1)+k_1+f(a_2)+k_2) \in (\bar{L}^f : (s, f(s)+k))$. So $f(c)+j \notin (L : f(s)+k)$ and $(f(c)+j)(y_1+y_2) \in (L : f(s)+k)$. Thus $y_1+y_2 \in (L : f(s)+k)^*$. Conversely, assume that $(L : f(s)+k)^*$ is an ideal of $f(A)+J$ and let $z_1 = (a_1, f(a_1)+k_1), z_2 = (a_2, f(a_2)+k_2) \in (\bar{L}^f : (s, f(s)+k))^*$. Then there exists $(b_1, f(b_1)+j_1),(b_2, f(b_2)+j_2), \notin (\bar{L}^f : (s, f(s)+k))$ such that $(a_1, f(a_1)+k_1)(b_1, f(b_1)+j_1),(a_2, f(a_2)+k_2)(b_2, f(b_2)+j_2) \in (\bar{L}^f : (s, f(s)+k))$. So $f(b_1)+j_1, f(b_2)+j_2 \notin (L : f(s)+k)$ and $(f(a_1)+k_1)(f(b_1)+j_1),(f(a_2)+k_2)(f(b_2)+j_2) \in (L : f(s)+k)$. Thus $f(a_1)+k_1, f(a_2)+k_2 \in (L : f(s)+k)^*$. So $f(a_1)+k_1+f(a_2)+k_2 \in (L : f(s)+k)^*$. Then there exists $f(c)+j \notin (L : f(s)+k)$ such that $(f(c)+j)(f(a_1)+k_1+f(a_2)+k_2) \in (L : f(s)+k)$. So $(c, f(c)+j) \notin (\bar{L}^f : (s, f(s)+k))$ and $(c, f(c)+j)(z_1+z_2) = (c, f(c)+j)(a_1+a_2, f(a_1)+k_1+f(a_2)+k_2) \in (\bar{L}^f : (s, f(s)+k))$. Then $z_1+z_2 \in (\bar{L}^f : (s, f(s)+k))^*$, as desired. □

Proof of Theorem 3.8

(1) $(a) \Rightarrow (b)$ Follows immediately from the fact that $S' \subseteq S \bowtie^f J$.
$(b) \Rightarrow (c)$ Assume that $I \bowtie^f J$ is an $(S \bowtie^f J)$-primal ideal of $A \bowtie^f J$. Then, there exists $(s, f(s)+k) \in S \bowtie^f J$ such that $(I \bowtie^f J : (s, f(s)+k))$ is a primal ideal of $A \bowtie^f J$. So by Lemma 3.9 $(I : s)^* \bowtie^f J = ((I : s) \bowtie^f J)^* = ((I \bowtie^f J : (s, f(s)+k)))^*$ is an ideal of $A \bowtie^f J$. Thus $(I : s)^*$ is an ideal of A and then I is a S-primal ideal of A.
$(c) \Rightarrow (a)$ Assume that I is an S-primal ideal of A, then there exists $s \in S$ such that $(I : s)$ is a primal ideal of A. Note that $I \cap S = \emptyset$ if and only if $(I \bowtie^f J) \cap S' = \emptyset$. Since $(I : s)^*$ is an ideal of A, then by Lemma 3.9, $(I \bowtie^f J : (s, f(s)))^* = ((I : s) \bowtie^f J)^* = (I : s)^* \bowtie^f J$ is an ideal of $A \bowtie^f J$. It follows that $I \bowtie^f J$ is an S'-primal ideal of $A \bowtie^f J$.

(2) First, note that $\bar{L}^f \cap (S \bowtie^f J) = \emptyset$ if and only if $L \cap (f(S)+J) = \emptyset$. Assume that $\bar{L}^f$ is an $(S \bowtie^f J)$-primal ideal of $A \bowtie^f J$. Then there exists $(s, f(s)+k) \in (S \bowtie^f J)$ such that $(\bar{L}^f : (s, f(s)+k))$ is a primal ideal of $A \bowtie^f J$. So $(\bar{L}^f : (s, f(s)+k))^*$ is an ideal of $A \bowtie^f J$. Then by Lemma 3.9, $(L : f(s)+k)^*$ is an ideal of $f(A)+J$ and then L is an $(f(S)+J)$-primal ideal of $f(A)+J$. Conversely, assume that L is an $(f(S)+J)$-primal ideal of $f(A)+J$. Then there exists $f(s)+k \in (f(S)+J)$ such that $(L : f(s)+k)^*$ is an ideal of $f(A)+J$. So by Lemma 3.9, $(\bar{L}^f : (s, f(s)+k))^*$ is an ideal of $A \bowtie^f J$. Thus $\bar{L}^f$ is an $(S \bowtie^f J)$-primal ideal of $A \bowtie^f J$. □

When S consists of units elements, we get the following corollary which is a consequence of Theorem 3.8.

Corollary 3.10. *I is a primal ideal of A if and only if $I \bowtie^f J$ is a primal ideal of $A \bowtie^f J$. In this case, $(I \bowtie^f J)^* = I^* \bowtie^f J$.*

The next corollary examines the case of the amalgamated duplication $(A = B, f = 1_A, S' = \{(s,s)/s \in S\}(\subseteq S \bowtie I))$.

Corollary 3.11. *Let K be a proper ideal of A. The following statements are equivalent:*

(a) *$K \bowtie I$ is an S'-primal ideal of $A \bowtie I$.*
(b) *$K \bowtie I$ is an $(S \bowtie I)$-primal ideal of $A \bowtie I$.*
(c) *K is an S-primal ideal of A.*

The next example illustrates Theorem 3.8 by providing a new class of S-primal ideal of $A \bowtie^f J$ that is not primal.

Example 3.12. *Let $A := \mathbb{Z}[X]$, where $\mathbb{Z}$ is the ring of integers, K be any ideal of A and $B := \frac{\mathbb{Z}[X]}{K}$. Let $f : A \to B$ be the canonical homomorphism and $S' = \{(2^n, \overline{2^n})/n \in \mathbb{N}\}$. Then $(2X) \bowtie^f J$ is an S'-primal ideal of $A \bowtie^f J$ that is not primal, for any proper ideal J of B.*

Proof. Since $((2X) : 2)^* = (X)^* = (X)$, then $(2X)$ is an S-primal ideal of A, where $S = \{2^n/n \in \mathbb{N}\}$. Then by Theorem 3.8, $(2X) \bowtie^f J$ is an S'-primal ideal of $A \bowtie^f J$. But $(2X) \bowtie^f J$ is a not primal ideal of $A \bowtie^f J$. Otherwise, by corollary 3.10 $(2X)^* \bowtie^f J = ((2X) \bowtie^f J)^*$ is an ideal of $A \bowtie^f J$ which is absurd since $(2X)^*$ is not an ideal of A. Indeed, $2 \in (2X)^*$ and $X \in (2X)^*$ but $2 + X \notin (2X)^*$ since $\overline{2+X} \notin Z(A/(2X))$.

□

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